A GENERALIZATION OF CONJECTURES OF BOGOMOLOV AND LANG OVER FINITELY GENERATED FIELDS

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Introduction

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and let $\overline{B}$ be a big polarization of K. Let A be an abelian variety over K, and L a symmetric ample line bundle on A. In the paper [2], we define the height pairing

$$\langle\ ,\ \rangle_L^{\overline{B}} : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$$

assigned to $\overline{B}$ and L with properties: $\langle x, x\rangle_L^{\overline{B}} \geq 0$ for all $x \in A(\overline{K})$ and the equality holds if and only if $x \in A(\overline{K})_{tor}$. For $x_1, \ldots, x_l \in A(\overline{K})$, we denote $\det\left(\langle x_i, x_j\rangle_L^{\overline{B}}\right)$ by $\delta_L^{\overline{B}}(x_1, \ldots, x_l)$. The purpose of this note is to prove the following theorem, which gives an answer of Poonen's question in [1].

Theorem A. *Let Γ be a subgroup of finite rank in $A(\overline{K})$, and X a subvariety of $A_{\overline{K}}$. Fix a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\Gamma \otimes \mathbb{Q}$. If the set $\{x \in X(\overline{K}) \mid \delta_L^{\overline{B}}(\gamma_1, \ldots, \gamma_n, x) \leq \epsilon\}$ is Zariski dense in X for every positive number ϵ, then X is a translation of an abelian subvariety of $A_{\overline{K}}$ by an element of Γ_{div}, where $\Gamma_{div} = \{x \in A(\overline{K}) \mid nx \in \Gamma$ for some positive integer $n\}$.*

In the case where $d = 0$, Poonen proved the equivalent result in [1]. Our argument for the proof of the above theorem essentially follows his ideas. A new point is that we remove measure-theoretical arguments from his original one, so that we can apply it to our case. Finally, we note that Theorem A substantially includes Lang's conjecture in the absolute form:

Lang's conjecture in the absolute form. *Let A be a complex abelian variety, Γ a subgroup of finite rank in $A(\mathbb{C})$, and X a subvariety of A. Then, there are abelian subvarieties $C_1, \ldots, C_n$ of A, and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that*

$$\overline{X(\mathbb{C}) \cap \Gamma} = \bigcup_{i=1}^{n} (C_i + \gamma_i) \quad \textit{and} \quad X(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{n} (C_i(\mathbb{C}) + \gamma_i) \cap \Gamma.$$

1. Review of arithmetic height functions over finitely generated fields

In this section, we give a quick review of arithmetic height functions over finitely generated fields. For details, see [2].

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a big polarization of K, i.e., B is a normal projective scheme over $\mathbb{Z}$, whose function field

Date: 18/August/1999, 4:45PM (JP), (Version 1.0).

is K, and $\overline{H}_1, \ldots, \overline{H}_d$ are nef and big C^∞-hermitian line bundles on B. For the definition of nef and big C^∞-hermitian line bundles, see [2, §2]. Let X be a projective variety over K and L a line bundle on X. Let us consider a C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) over B. Namely, $\mathcal{X}$ is a projective integral scheme over B, whose generic fiber over B is X, and $\overline{\mathcal{L}}$ is a C^∞-hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}$, which gives rise to L on the generic fiber of $\mathcal{X} \to B$. For $x \in X(\overline{K})$, let Δ_x be the closure of the image $\operatorname{Spec}(\overline{K}) \overset{x}{\longrightarrow} X \hookrightarrow \mathcal{X}$. Then, we define the height of x with respect to the polarization $\overline{B}$ and the C^∞-model $(\mathcal{X}, \overline{\mathcal{L}})$ to be

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) = \frac{1}{[K(x):K]} \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}\big|_{\Delta_x}) \cdot \widehat{c}_1(\pi^*(\overline{H}_1)\big|_{\Delta_x}) \cdots \widehat{c}_1(\pi^*(\overline{H}_d)\big|_{\Delta_x}) \right),$$

where $\pi : \mathcal{X} \to B$ is the canonical morphism. If $(\mathcal{X}', \overline{\mathcal{L}}')$ is another C^∞-model of (X, L), then there is a constant C such that

$$|h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(x) - h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(x)| \leq C$$

for all $x \in X(\overline{K})$. Thus, modulo the set of bounded functions, we can assign the unique height function $h^{\overline{B}}_L : X(\overline{K}) \to \mathbb{R}$ to $\overline{B}$ and L. Note that if $\sigma \in \operatorname{Gal}(\overline{K}/K)$, then $\Delta_x = \Delta_{\sigma(x)}$. Thus, $h^{\overline{B}}_L(\sigma(x)) = h^{\overline{B}}_L(x)$. The first important theorem is the following Northcott's theorem for our height functions.

Theorem 1.1 ([2, Theorem 4.3])**.** *If L is ample, then, for any numbers M and any positive integers e, the set*

$$\left\{ x \in X(\overline{K}) \mid h^{\overline{B}}_L(x) \leq M, \quad [K(x):K] \leq e \right\}$$

is finite.

Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Then, as the usual height functions over a number field, there is the canonical height function $\hat{h}^{\overline{B}}_L$. This gives rise to a quadric form on $A(\overline{K})$, so that if we set

$$\langle x, y \rangle^{\overline{B}}_L = \frac{1}{2}\left(\hat{h}^{\overline{B}}_L(x+y) - \hat{h}^{\overline{B}}_L(x) - \hat{h}^{\overline{B}}_L(y) \right)$$

for $x, y \in A(\overline{K})$, then $\langle\ ,\ \rangle^{\overline{B}}_L$ is a bi-linear form on $A(\overline{K})$. Concerning this bi-linear form, we have the following.

Proposition 1.2 ([2, §§3.4])**.** (1) *$\langle x, x \rangle^{\overline{B}}_L \geq 0$ for all $x \in A(\overline{K})$, and the equality holds if and only if x is a torsion point. Namely, $\langle\ ,\ \rangle^{\overline{B}}_L$ is positive definite on $A(\overline{K}) \otimes \mathbb{Q}$.*

(2) *If $f : A \to A'$ is a homomorphism of abelian varieties over K, and L' is a symmetric ample line bundle on A', then there is a positive number a with*

$$\langle f(x), f(x) \rangle^{\overline{B}}_{L'} \leq a \langle x, x \rangle^{\overline{B}}_L$$

for all $x \in A(\overline{K})$.

Remark 1.3. (2) of Proposition 1.2 holds even if f, A' and L' are not defined over K. Let K' be a finite extension field of K such that f, A' and L' are defined over K'. Let

$\phi : B^{K'} \to B$ be the normalization of B in K'. Then, $\overline{B}^{K'} = (B^{K'}; \phi^*(\overline{H}_1), \dots, \phi^*(\overline{H}_d))$ gives rise to a big polarization of K'. Thus, there is a positive number a' with

$$\langle f(x), f(x) \rangle_{L'}^{\overline{B}^{K'}} \leq a' \langle x, x \rangle_{L}^{\overline{B}^{K'}}$$

for all $x \in A(\overline{K})$. On the other hand, $\langle\ ,\ \rangle_{L}^{\overline{B}^{K'}} = [K' : K] \langle\ ,\ \rangle_{L}^{\overline{B}}$. Hence,

$$\langle f(x), f(x) \rangle_{L'}^{\overline{B}^{K'}} \leq a' [K' : K] \langle x, x \rangle_{L}^{\overline{B}}$$

for all $x \in A(\overline{K})$.

The crucial result for this note is the following solution of Bogomolov's conjecture over finitely generated fields, which is a generalization of [3] and [4].

Theorem 1.4 ([2, Theorem 8.1])**.** *Let X be a subvariety of $A_{\overline{K}}$. If the set*

$$\{x \in X(\overline{K}) \mid \hat{h}_L^{\overline{B}}(x) \leq \epsilon\}$$

is Zariski dense in X for every positive number ϵ, then X is a translation of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

2. Small points with respect to a group of finite rank

The contexts in this section are essentially due to Poonen [1]. We just deal with his ideas in a general situation.

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$, and let $\overline{B}$ be a big polarization of K. Let A be an abelian variety over K, and L a symmetric ample line bundle on A. Let

$$\langle\ ,\ \rangle_L^{\overline{B}} : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$$

be the height pairing associated with $\overline{B}$ and L as in §1. Let Γ be a subgroup of finite rank in $A(\overline{K})$. A non-empty subset S of $A(\overline{K})$ is said to be *small with respect to* Γ if there is a decomposition $s = \gamma(s) + z(s)$ for each $s \in S$ with the following properties:

(a) $\gamma(s) \in \Gamma$ for all $s \in S$.
(b) For any $\epsilon > 0$, there is a finite proper subset S' of S such that $\langle z(s), z(s) \rangle_L^{\overline{B}} \leq \epsilon$ for all $s \in S \setminus S'$.

Especially a small subset S with respect to $\{0\}$ is said to be *small*. Namely, a non-empty subset S of $A(\overline{K})$ is small if and only if, for any positive numbers ϵ, there is a finite proper subset S' of S with $\langle x, x \rangle_L^{\overline{B}} \leq \epsilon$ for all $s \in S \setminus S'$. Note that in the above definition, S' is proper, i.e., $S \setminus S' \neq \emptyset$. Let us begin with the following proposition.

Proposition 2.1. *Let S be a non-empty subset of $A(\overline{K})$ and Γ a subgroup of finite rank in $A(\overline{K})$. Then, we have the following:*

(1) *If S is small with respect to Γ, then any infinite subsets of S are small with respect to Γ.*
(2) *We assume that S is finite. Then S is small (with respect to $\{0\}$) if and only if S contains a torsion point.*
(3) *We assume that S is infinite. Let N be a positive integer, and $[N]$ an endomorphism of A given by $[N](x) = Nx$. If S is small with respect to Γ, then so is $[N](S)$.*

(4) *Let* $\{x_n\}$ *be a sequence in* $A(\overline{K})$ *with the following properties:*

(4.1) *If* $n \neq m$*, then* $x_n \neq x_m$*.*

(4.2) *Each* x_n *has a decomposition* $x_n = \gamma_n + y_n$ *with* $\gamma_n \in \Gamma$*.*

(4.3) $\lim_{n\to\infty} \langle y_n, y_n\rangle_L^{\overline{B}} = 0$.

Then, $\{x_n \mid n = 1, 2, \dots\}$ *is small with respect to* Γ*.*

Proof. (1) and (4) are obvious.

(2) Clearly, if S contains a torsion point, then S is small. We assume that S is small. We set $\lambda = \min\{\langle s, s\rangle_L^{\overline{B}} \mid s \in S\}$. If $\lambda > 0$, then there is a finite proper subset S' of S such that $\langle s, s\rangle_L^{\overline{B}} < \lambda$ for all $s \in S \setminus S'$. This is a contradiction. Thus, $\lambda = 0$, which means that S contains a torsion point.

(3) We fix a map $t : [N](S) \to S$ with $[N](t(s)) = s$ for all $s \in [N](S)$. Then, we have a decomposition $s = [N](\gamma(t(s))) + [N](z(t(s)))$ for each $s \in [N](S)$. Clearly (a) in the definition of small sets is satisfied. Let ϵ be an arbitrary positive number. Then, there is a finite subset T of S such that $\langle z(s), z(s)\rangle_L^{\overline{B}} \leq \epsilon/N^2$ for all $s \in S \setminus T$. If we set $T' = \{s \in [N](S) \mid t(s) \in T\}$, then T' is finite. Moreover, $\langle [N](z(t(s))), [N](z(t(s)))\rangle_L^{\overline{B}} \leq \epsilon$ for all $s \in [N](S) \setminus T'$. Therefore, we have (b) in the definition of small sets. □

Moreover, we have the following, which is a consequence of Bogomolov's conjecture.

Theorem 2.2. *Let* S *be a small set of* $A(\overline{K})$*, i.e.,* S *is small with respect to* $\{0\}$*. Then, there are abelian subvarieties* $C_1, \dots, C_n$*, torsion points* $c_1, \dots, c_n$*, and finite non-torsion points* $b_1, \dots, b_m$ *such that*

$$\overline{S} = \bigcup_{i=1}^{n} (C_i + c_i) \cup \{b_1, \dots, b_m\},$$

where $\overline{S}$ *is the Zariski closure of* S*.*

Proof. It is sufficient to show that a positive dimensional irreducible component X of $\overline{S}$ is a translation of an abelian subvariety of A by a torsion point. Let S' be the set of points in S, which is contained in $X(\overline{K})$. Then, the Zariski closure of S' is X. In particular, S' is infinite set, so that S' is small. Thus, X is a translation of an abelian subvariety of A by a torsion point by virtue of Theorem 1.4. □

Let S be a small subset with respect to Γ. For each $n \geq 2$, let us consider a homomorphism $\beta_n : A^n \to A^{n-1}$ given by $\beta_n(a_1, \dots a_n) = (a_2 - a_1, a_3 - a_1, \dots, a_n - a_1)$. Let F be a finite extension field of K in $\overline{K}$. For $x \in A(\overline{K})$, we denote by $O_F(x)$ the orbit of x by the Galois group $\mathrm{Gal}(\overline{K}/F)$. Noting $O_F(x)^n \subseteq A(\overline{K})^n$, for a subset T of S, we define $\mathcal{D}_n(T, F)$ to be

$$\mathcal{D}_n(T, F) = \bigcup_{s\in T} \beta_n(O_F(s)^n).$$

We denote the Zariski closure of $\mathcal{D}_n(T, F)$ by $\overline{\mathcal{D}}_n(T, F)$. On A^n, we can give the height pairing associated with $\bigotimes_{i=1}^n p_i^*(L)$ and $\overline{B}$, where $p_i : A^n \to A$ is the projection to the i-th factor. By abuse of notation, we denote this by $\langle\ ,\ \rangle_L^{\overline{B}}$.

Proposition 2.3. *Let $f : A \to A'$ be a homomorphism of abelian varieties over $\overline{K}$. Let F be a finite extension field of K in $\overline{K}$. We assume that there is a finitely generated subgroup Γ_0 of Γ such that $\Gamma_0 \subseteq A(K)$ and $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$. Then, we have the following:*

(1) *$f^{n-1}(\mathcal{D}_n(S, F))$ is small (with respect to $\{0\}$), where $f^{n-1} : A^{n-1} \to A'^{n-1}$ is the morphism given by $f^{n-1}(x_1, \dots, x_{n-1}) = (f(x_1), \dots, f(x_{n-1}))$.*

(2) *Let $b_1, \dots, b_l$ be non-torsion points in $f^{n-1}(\mathcal{D}_n(S, F))$. Then, there is a finite proper subset S' of S such that $b_i \notin f^{n-1}(\mathcal{D}_n(S \setminus S', F))$ for all i.*

Proof. Let σ, τ be elements of $\mathrm{Gal}(\overline{K}/F)$. Then, $\sigma(\gamma(s)) - \tau(\gamma(s))$ is torsion because $n\gamma(s) \in \Gamma_0$ for some $n > 0$. Thus,

$$\|\sigma(s) - \tau(s)\|_L^{\overline{B}} = \|\sigma(z(s)) - \tau(z(s))\|_L^{\overline{B}} \leq 2\|z(s)\|_L^{\overline{B}},$$

where $\|x\|_L^{\overline{B}} = \sqrt{\langle x, x\rangle_L^{\overline{B}}}$. Therefore,

$$\|\beta_n(x)\|_L^{\overline{B}} \leq 2\sqrt{n-1}\|z(s)\|_L^{\overline{B}}$$

for all $x \in O_F(s)^n$. Let L' be a symmetric ample line bundle on A'. Then, by (2) of Proposition 1.2 (or Remark 1.3), there is a positive constant a with $\langle f(x), f(x)\rangle_{L'}^{\overline{B}} \leq a\langle x, x\rangle_L^{\overline{B}}$ for all $x \in A(\overline{K})$. Thus,

$$\|f^{n-1}(\beta_n(x))\|_{L'}^{\overline{B}} \leq 2\sqrt{a(n-1)}\|z(s)\|_L^{\overline{B}} \tag{2.3.1}$$

for all $x \in O_F(s)^n$.

First, let us see (2). We set $\mu = \min\{\|b_i\|_{L'}^{\overline{B}} \mid i = 1, \dots, l\} > 0$. Then there is a finite proper subset S' of S with

$$\|z(s)\|_L^{\overline{B}} < \frac{\mu}{2\sqrt{a(n-1)}}$$

for all $s \in S \setminus S'$. Thus, by (2.3.1),

$$\|f^{n-1}(\beta_n(x))\|_{L'}^{\overline{B}} < \mu$$

for all $x \in \bigcup_{s \in S \setminus S'} O_F(s)^n$. Hence, $b_i \notin f^{n-1}(\mathcal{D}_n(S \setminus S', F))$ for all i.

Next we consider (1). If $f^{n-1}(\mathcal{D}_n(S, F))$ is infinite, then the assertion of (1) is obvious by (2.3.1). Otherwise, let $\{b_1, \dots, b_n\}$ be the set of all non-torsion points in $f^{n-1}(\mathcal{D}_n(S, F))$. Then, by (2), we can find a finite proper subset S' of S with

$$\emptyset \neq f^{n-1}(\mathcal{D}_n(S \setminus S', F)) \subseteq f^{n-1}(\mathcal{D}_n(S, F)) \setminus \{b_1, \dots, b_n\}.$$

Hence $f^{n-1}(\mathcal{D}_n(S, F))$ contains a torsion point. Therefore, $f^{n-1}(\mathcal{D}_n(S, F))$ is small. □

Let S be a small subset with respect to Γ. From now on, we assume the following:

(A) S is infinite.

(B) There is a finitely generated subgroup Γ_0 of Γ such that $\Gamma_0 \subseteq A(K)$ and $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$.

Let F be a finite extension field of K in $\overline{K}$. A pair (S, F) is said to be *n-minimized* if the following properties are satisfied:

(i) $\overline{\mathcal{D}}_n(S', F') = \overline{\mathcal{D}}_n(S, F)$ for any infinite subsets S' of S and any finite extension fields F' of F in $\overline{K}$. (Recall that $\overline{\mathcal{D}}_n(\cdot, \cdot)$ is the Zariski closure of $\mathcal{D}_n(\cdot, \cdot)$.)

(ii) $\overline{\mathcal{D}}_n([N](S), F) = \overline{\mathcal{D}}_n(S, F)$ for any positive integers N.

Note that $[N](O_F(s)) = O_F([N](s))$ for $s \in S$ and a positive integer N, so that $\mathcal{D}_n([N](S), F) = [N](\mathcal{D}_n(S, F))$. Therefore, (ii) is equivalent to saying that $[N](\overline{\mathcal{D}}_n(S, F)) = \overline{\mathcal{D}}_n(S, F)$ for any positive integers N. First let us consider the following proposition.

Proposition 2.4. (1) *If we fix $n \geq 2$, then there are an infinite subset T of S, a positive integer N, and a finite extension field F of K in $\overline{K}$ such that $([N](T), F)$ is n-minimized.*

(2) *Let F be a finite extension field of K in $\overline{K}$. Let N be a positive integer, S' an infinite subset of $[N](S)$, and F' a finite extension field of F in $\overline{K}$. If (S, F) is n-minimized, then $\overline{\mathcal{D}}_n(S', F') = \overline{\mathcal{D}}_n(S, F)$.*

Proof. (1) Let F be a finite extension field of K in $\overline{K}$. A pair (S, F) is said to be *weakly n-minimized* if the above property (i) is satisfied. First, we claim the following.

Claim 2.4.1. (a) *If we fix $n \geq 2$, then there are an infinite subset T of S and a finite extension field F of K such that (T, F) is weakly n-minimized.*

(b) *Let F be a finite extension field of K in $\overline{K}$. If (S, F) is weakly n-minimized, then there are abelian subvarieties $C_1, \dots, C_n$, and torsion points $c_1, \dots, c_n$ such that*

$$\overline{\mathcal{D}}_n(S, F) = \bigcup_{i=1}^{n} (C_i + c_i).$$

(c) *Let F be a finite extension field of K in $\overline{K}$, and N a positive integer. If (S, F) is weakly n-minimized, then so is $([N](S), F)$.*

(a) This is obvious by Noetherian induction.

(b) By Theorem 2.2, there are abelian subvarieties $C_1, \dots, C_n$, torsion points $c_1, \dots, c_n$, and finite non-torsion points $b_1, \dots, b_m$ such that

$$\overline{\mathcal{D}}_n(S, F) = \bigcup_{i=1}^{n} (C_i + c_i) \cup \{b_1, \dots, b_m\}.$$

By virtue of (2) of Proposition 2.3, we can find a finite set T of S such that

$$\overline{\mathcal{D}}_n(S \setminus T, K) \subseteq \bigcup_{i=1}^{n} (C_i + c_i) \subseteq \overline{\mathcal{D}}_n(S, F).$$

Here, $\overline{\mathcal{D}}_n(S \setminus T, K) = \overline{\mathcal{D}}_n(S, K)$. Thus, we get (b).

(c) Let S_1 be an infinite subset of $[N](S)$ and F' a finite extension field of F in $\overline{K}$. We take a subset S' of S with $[N](S') = S_1$. Then, $\overline{\mathcal{D}}_n(S', F') = \overline{\mathcal{D}}_n(S, F)$. Thus, since $[N]$ is a finite and surjective morphism, we can see

$$\overline{\mathcal{D}}_n(S_1, F') = \overline{\mathcal{D}}_n([N](S'), F') = [N](\overline{\mathcal{D}}_n(S', F')) = [N](\overline{\mathcal{D}}_n(S, F)) = \overline{\mathcal{D}}_n([N](S), F).$$

Hence, we have (c).

Let us start the proof of (1). By virtue of (a), there are an infinite subset T of S and a finite extension field F of K such that (T, F) is weakly n-minimized. Hence, by (b), there

are abelian subvarieties $C_1, \ldots, C_n$, and torsion points $c_1, \ldots, c_n$ such that

$$\overline{\mathcal{D}}_n(T, F) = \bigcup_{i=1}^{n} (C_i + c_i).$$

Let N be a positive integer with $Nc_i = 0$ for all i. Then,

$$\overline{\mathcal{D}}_n([N](T), F) = [N](\overline{\mathcal{D}}_n(T, F)) = \bigcup_{i=1}^{n} C_i.$$

Here we claim that $([N](T), F)$ is n-minimized. By (c), $([N](T), F)$ is weakly n-minimized. Moreover, for any positive integers N',

$$\begin{aligned} \overline{\mathcal{D}}_n([N']([N](T)), F) &= [N'](\overline{\mathcal{D}}_n([N](T), F)) \\ &= [N'] \left(\bigcup_{i=1}^{n} C_i \right) = \bigcup_{i=1}^{n} C_i \\ &= \overline{\mathcal{D}}_n([N](T), F). \end{aligned}$$

Thus, $([N](T), F)$ is n-minimized.

(2) Let N be a positive integer, S' an infinite subset of $[N](S)$, and F' a finite extension field of F. By (c), $([N](S), F)$ is weakly n-minimized. Thus,

$$\overline{\mathcal{D}}_n(S', F') = \overline{\mathcal{D}}_n([N](S), F) = \overline{\mathcal{D}}_n(S, F).$$

Therefore, we get (2). $\square$

Finally, let us consider the following theorem, which is crucial for our note.

Theorem 2.5. *Let F be a finite extension field of K in $\overline{K}$. Then, the following* (1), (2) *and* (3) *are equivalent.*

(1) (S, F) *is n-minimized for all $n \geq 2$.*
(2) (S, F) *is n-minimized for some $n \geq 2$.*
(3) (S, F) *is 2-minimized.*

Moreover, under the above equivalent conditions, there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_n(S, F) = C^{n-1}$ for all $n \geq 2$.

Proof. Let us begin with the following two lemmas.

Lemma 2.6. *Let F be a finite extension field of K in $\overline{K}$, and C an abelian subscheme of A_F over F. We assume that there is a positive integer e with the following property: For each $s \in S$, there is a subset $T(s)$ of $O_F(s) \times O_F(s)$ such that $\beta_2(T(s)) \subseteq C(\overline{K})$ and $\#(T(s)) \geq \#(O_F(s) \times O_F(s))/e$. Then, there is a finite subset S' of S and a positive integer N with $\mathcal{D}_2([N](S \setminus S'), F) \subseteq C(\overline{K})$.*

Proof. Let $\pi : A \to A/C$ be a natural homomorphism. Fix $s \in S$. Let F' be a finite Galois extension of F such that F' contains $F(s)$. Then, there is a natural surjective map

$$\phi : \mathrm{Gal}(F'/F) \to O_F(s),$$

whose fibers are cosets of the stabilizer of s. If we set $E(s) = (\phi \times \phi)^{-1}(T(s))$, then $\#(E(s)) \geq \#(\mathrm{Gal}(F'/F) \times \mathrm{Gal}(F'/F))/e$ and $\sigma(\pi(s)) = \tau(\pi(s))$ for all $(\sigma, \tau) \in E(s)$. Let $G_{\pi(s)}$ be

the stabilizer of $\pi(s)$ by the action of $\mathrm{Gal}(F'/F)$, and let R be the set of all $(\sigma, \tau) \in \mathrm{Gal}(F'/F) \times \mathrm{Gal}(F'/F)$ with $\sigma(\pi(s)) = \tau(\pi(s))$. Then, we have

$$\#(R) = \#(G_{\pi(s)})\#(\mathrm{Gal}(F'/F)) \quad \text{and} \quad \#(R) \geq \frac{\#(\mathrm{Gal}(F'/F) \times \mathrm{Gal}(F'/F))}{e}.$$

Thus, $[\mathrm{Gal}(F'/F) : G_{\pi(s)}] \leq e$, which means that $[F(\pi(s)) : F] \leq e$. Then, since $\pi(\mathcal{D}_2(S, F))$ is small, by virtue of Northcott's theorem (cf. Theorem 1.1), $\pi(\mathcal{D}_2(S, F))$ is finite. By (2) of Proposition 2.3, there is a finite proper subset S' of S such that $\pi(\mathcal{D}_2(S \setminus S', F))$ consists of torsion points. Hence, there is a positive integer N such that $[N](\pi(\mathcal{D}_2(S \setminus S', F))) = \{0\}$. Therefore, $\mathcal{D}_2([N](S \setminus S'), F) \subseteq C(\overline{K})$. □

Lemma 2.7. *Let F be a finite extension field of K in $\overline{K}$. If (S, F) are 2-minimized, then there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_n(S, F) = C^{n-1}$ for all $n \geq 2$.*

Proof. First, let us consider the case $n = 2$. By using (b) of Claim 2.4.1, we can find abelian subvarieties $C_1, \ldots, C_e$ with

$$\overline{\mathcal{D}}_2(S, F) = \bigcup_{i=1}^{e} C_i$$

because $\overline{\mathcal{D}}_2(S, F)$ is stable by the endomorphism $[N]$ for every positive integer N. Thus, in order to see $e = 1$, it is sufficient to find C_i, a positive integer N_1, an infinite subset S_1 of S, and a finite extension field F_1 of F such that

$$\mathcal{D}_2([N_1](S_1), F_1) \subseteq C_i(\overline{K}).$$

Let F_1 be a finite extension field of F such that C_i's are defined over F_1. For each $s \in S$, let $T_i(s)$ be the set of all elements $x \in O_{F_1}(s)^2$ with $\beta_2(x) \in C_i(\overline{K})$. We choose a map $\lambda : S \to \{1, \ldots, e\}$ such that $\#(T_{\lambda(s)}(s))$ gives rise to the maximal value in $\{\#(T_i(s)) \mid i = 1, \ldots, e\}$. By using the pigeonhole principle, there are $i \in \{1, \ldots, e\}$ and an infinite subset S' of S with $\lambda(s) = i$ for all $s \in S'$. Then, for all $s \in S'$, $\beta_2(T_i(s)) \subseteq C_i(\overline{K})$ and $\#(T_i(s)) \geq \#(O_{F_1}(s)^2)/e$. Thus, by Lemma 2.6, there are an infinite subset S_1 of S' and a positive integer N_1 with $\mathcal{D}_2([N_1](S_1), F_1) \subseteq C_i(\overline{K})$.

From now on, we denote C_i by C. Then, $\overline{\mathcal{D}}_2(S, F) = C$. Let us try to see $\overline{\mathcal{D}}_n(S, F) = C^{n-1}$ for all $n \geq 2$. Clearly, $\overline{\mathcal{D}}_n(S, F) \subseteq C^{n-1}$. Thus it is sufficient to find a positive integer N_2, an infinite subset S_2 of S, and a finite extension field F_2 of F such that

$$\overline{\mathcal{D}}_n([N_2](S_2), F_2) = C^{n-1}.$$

By (1) of Proposition 2.4, there are a positive integer N_2, an infinite subset S_2 of S and a finite extension field F_2 of F such that $([N_2](S_2), F_2)$ is n-minimized. Thus, as before, there are abelian subvarieties $G_1, \ldots, G_l$ with $\overline{\mathcal{D}}_n([N_2](S_2), F_2) = \bigcup_{j=1}^{l} G_j$. Moreover, replacing F_2 by a finite extension field of F_2, we may assume that C and G_j's are defined over F_2. On this stage, we would like to show that

$$\overline{\mathcal{D}}_n([N_2](S_2), F_2) = C^{n-1}.$$

In the same way as before, we can find G_j, say G, and an infinite subset S' of $[N_2](S_2)$ such that for all $s \in S'$, there is a subset $T(s)$ of $O_{F_2}(s)^n$ with $\#(T(s)) \geq \#(O_{F_2}(s)^n)/l$ and $\beta_n(T(s)) \subseteq G(\overline{K})$. Let $C^{(q)} = 0 \times \cdots \times C \times \cdots \times 0$ be the q-th factor of C^{n-1}, and

$G^{(q)} = G \cap C^{(q)}$ for $1 \leq q \leq n-1$. Since $G \subseteq C^{n-1}$, it is sufficient to see the following claim to conclude the proof of our lemma.

Claim 2.7.1. *$G^{(q)} = C^{(q)}$ for each $1 \leq q \leq n-1$.*

For each $t_1, \ldots, t_q, t_{q+2}, \ldots, t_n \in O_{F_2}(s)$, we set

$$J(t_1, \ldots, t_q, t_{q+2}, \ldots, t_n) = \{x \in O_{F_2}(s) \mid (t_1, \ldots, t_q, x, t_{q+2}, \ldots, t_n) \in T(s)\}.$$

We choose $s_1, \ldots, s_q, s_{q+2}, \ldots, s_n \in O_{F_2}(s)$ such that $\#(J(s_1, \ldots, s_q, s_{q+2}, \ldots, s_n))$ is maximal among $\{\#(J(t_1, \ldots, t_q, t_{q+2}, \ldots, t_n)) \mid t_1, \ldots, t_q, t_{q+2}, \ldots, t_n \in O_{F_2}(s)\}$. Then,

$$\#(J(s_1, \ldots, s_q, s_{q+2}, \ldots, s_n))\#(O_{F_2}(s)^{n-1}) \geq \#(T(s)) \geq \frac{\#(O_{F_2}(s)^n)}{l}.$$

Thus if we set $L(s) = J(s_1, \ldots, s_q, s_{q+2}, \ldots, s_n)$, then $\#(L(s)) \geq \#(O_{F_2}(s))/l$ and

$$\beta_n(s_1, \ldots, s_q, x, s_{q+2}, \ldots, s_n) \in G(\overline{K})$$

for all $x \in L(s)$. Therefore, for all $(x, x') \in L(s) \times L(s)$,

$$\beta_n(0, \ldots, 0, x - x', 0, \ldots, 0) = \\ \beta_n(s_1, \ldots, s_q, x, s_{q+2}, \ldots, s_n) - \beta_n(s_1, \ldots, s_q, x, s_{q+2}, \ldots, s_n) \in G(\overline{K}).$$

This means that $\beta_2(x, x') \in G^{(q)}(\overline{K})$ for all $(x, x') \in L(s) \times L(s)$ if we view $G^{(q)}$ as a subscheme of A. Here $\#(L(s) \times L(s)) \geq \#(O_{F_2}(s) \times O_{F_2}(s))/l^2$. By Lemma 2.6, there are an infinite subset S'' of S' and a positive integer N'' with $\overline{\mathcal{D}}_2([N''](S''), F_2) \subseteq G^{(q)}$, which implies that $G^{(q)} = C^{(q)}$ because $\overline{\mathcal{D}}_2([N''](S''), F_2) = C$ by (2) of Proposition 2.4. □

Let us start the proof of Theorem 2.5. The last assertion is nothing more than Lemma 2.7, so that it is sufficient to show that (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1).

(2) $\Longrightarrow$ (3): By (1) of Proposition 2.4, there are an infinite subset T of S, a positive integer N_1, and a finite extension field F_1 of F in $\overline{K}$ such that $([N_1](T), F_1)$ is 2-minimized. Then, by Lemma 2.7, there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_2([N_1](T), F_1) = C$ and $\overline{\mathcal{D}}_n([N_1](T), F_1) = C^{n-1}$. Thus, $\overline{\mathcal{D}}_n(S, F) = C^{n-1}$ because (S, F) is n-minimized. For all $x, x' \in O_F(s)$ with $s \in S$,

$$\beta_n(s, x, s, \ldots, s) - \beta_n(s, x', s, \ldots, s) = (x - x', 0, \ldots, 0) \in C(\overline{K})^{n-1}.$$

Thus, $\beta_2(O_F(s)^2) \subseteq C(\overline{K})$ for all $s \in S$. Therefore, $\overline{\mathcal{D}}_2(S, F) \subseteq C$. Let S' be an infinite subset of S, and F' a finite extension field of K. In order to see that $\overline{\mathcal{D}}_2(S', F') = C$, we may assume that $S' \subseteq T$ and $F_1 \subseteq F'$. Then,

$$[N_1](\overline{\mathcal{D}}_2(S', F')) = \overline{\mathcal{D}}_2([N_1](S'), F') = \overline{\mathcal{D}}_2([N_1](T), F_1) = C.$$

Thus, $\overline{\mathcal{D}}_2(S', F') = C$ because $\overline{\mathcal{D}}_2(S', F') \subseteq C$. Hence (S, F) satisfies the property (i) in the definition of "2-minimized". Moreover, $[N](\overline{\mathcal{D}}_2(S, F)) = [N](C) = C$ for all positive integers N. Therefore, (S, F) is 2-minimized.

(3) $\Longrightarrow$ (1): By Lemma 2.7, there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_n(S, F) = C^{n-1}$ for all $n \geq 2$. Fix $n \geq 2$. By (1) of Proposition 2.4, there are an infinite subset T of S, a positive integer N_1, and a finite extension field F_1 of F in $\overline{K}$ such that

$([N_1](T), F_1)$ is n-minimized. Since $([N_1](T), F_1)$ is 2-minimized and $\overline{\mathcal{D}}_2([N_1](T), F_1) = C$, we have $\overline{\mathcal{D}}_n([N_1](T), F_1) = C^{n-1}$ by Lemma 2.7. Thus, as before, we can see that (S, F) is n-minimized. $\square$

3. Proof of Theorem A

3.1. **Preliminary of linear algebra.** Let V be a vector space over $\mathbb{R}$, and $\langle\ ,\ \rangle$ an inner product on V. For a finite set of linearly independent vectors $\Lambda = \{v_1, \ldots, v_n\}$, we define

$$\Delta_\Lambda : V \times V \to \mathbb{R}$$

to be

$$\Delta_\Lambda(x, y) = \det \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_n\rangle & \langle v_1, y\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_n, v_1\rangle & \cdots & \langle v_n, v_n\rangle & \langle v_n, y\rangle \\ \langle x, v_1\rangle & \cdots & \langle x, v_n\rangle & \langle x, y\rangle \end{pmatrix}$$

Then, we have the following:

Proposition 3.1.1. (1) *Δ_Λ is a bi-linear map.*
(2) *Δ_Λ is symmetric and positive semidefinite.*
(3) *For all $v \in \operatorname{Span}(\Lambda)$ and $x \in V$, $\Delta_\Lambda(v, x) = 0$.*
(4) *If $\Lambda' = \{v'_1, \ldots, v'_n\}$ is another finite set of linearly independent vectors with $\operatorname{Span}(\Lambda') = \operatorname{Span}(\Lambda)$, then*

$$\Delta_{\Lambda'} = \frac{\det(\Lambda')}{\det(\Lambda)} \Delta_\Lambda,$$

where $\det(\Lambda) = \det(\langle v_i, v_j\rangle)$ and $\det(\Lambda') = \det(\langle v'_i, v'_j\rangle)$.
(5) *There are linear maps $p_\Lambda : V \to \operatorname{Span}(\Lambda)$ and $q_\Lambda : V \to \operatorname{Span}(\Lambda)^\perp$ with $x = p_\Lambda(x) + q_\Lambda(x)$ for all $x \in V$, where $\operatorname{Span}(\Lambda)^\perp = \{x \in V \mid \langle x, v\rangle = 0$ for all $v \in \operatorname{Span}(\Lambda)\}$.*
(6) *$\Delta_\Lambda(x, x) = \det(\Lambda)\langle q_\Lambda(x), q_\Lambda(x)\rangle$ for all $x \in V$. In particular, $\Delta_\Lambda(x, x) \le \det(\Lambda)\langle x, x\rangle$ and the equality holds if and only if $x \in \operatorname{Span}(\Lambda)^\perp$.*

Proof. (1), (2) and (3) are straightforward from the definition of Δ_Λ.

(4) First of all, there is an invertible matrix P with $(v'_1, \ldots, v'_n) = (v_1, \ldots, v_n)P$. Then it is easy to see that $(\langle v'_i, v'_j\rangle) = P(\langle v_i, v_j\rangle)^t P$. Thus, $\det(\Lambda') = \det(P)^2 \det(\Lambda)$. On the other hand, since

$$(v'_1, \ldots, v'_n, x) = (v_1, \ldots, v_n, x) \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix},$$

in the same way as above, we have $\Delta_{\Lambda'}(x, x) = \det(P)^2 \Delta_\Lambda(x, x)$. Therefore,

$$\Delta_{\Lambda'}(x, y) = \frac{\det(\Lambda')}{\det(\Lambda)} \Delta_\Lambda(x, y)$$

for all $x, y \in V$ because $2\Delta_\Lambda(x, y) = \Delta_\Lambda(x + y, x + y) - \Delta_\Lambda(x, x) - \Delta_\Lambda(y, y)$.

(5) For $x \in V$, solving the equation

$$\sum_{j=1}^{n} \lambda_j \langle v_j, v_i \rangle = \langle x, v_i \rangle \quad \text{for all } i = 1, \ldots, n,$$

we can find a unique vector $v = \sum \lambda_j v_j \in \operatorname{Span}(\Lambda)$ such that $x - v$ is perpendicular to $\operatorname{Span}(\Lambda)$. Thus, if we denote the vector v by $p_\Lambda(x)$ and the vector $x - v$ by $q_\Lambda(x)$, then we have (5).

(6) Using (1), (2), (3) and (5), we can see

$$\begin{aligned}\Delta_\Lambda(x, x) &= \Delta_\Lambda(p_\Lambda(x), p_\Lambda(x)) + 2\Delta_\Lambda(p_\Lambda(x), q_\Lambda(x)) + \Delta_\Lambda(q_\Lambda(x), q_\Lambda(x)) \\ &= \Delta_\Lambda(q_\Lambda(x), q_\Lambda(x)) = \det(\Lambda)\langle q_\Lambda(x), q_\Lambda(x)\rangle.\end{aligned}$$

□

Corollary 3.1.2. *Let $f : V \to V'$ be a linear map of vector spaces over $\mathbb{R}$, and let $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ be inner products of V and V' respectively. We assume that there is a positive constant a with $\langle f(x), f(x)\rangle' \leq a\langle x, x\rangle$ for all $x \in V$. Let $\Lambda = \{v_1, \ldots, v_n\}$ be a set of linearly independent vectors in V, and $\Lambda' = \{v'_1, \ldots, v'_{n'}\}$ a basis of $f(\operatorname{Span}(\Lambda))$. Then, for all $x \in V$,*

$$\Delta_{\Lambda'}(f(x), f(x)) \leq a\frac{\det(\Lambda')}{\det(\Lambda)}\Delta_\Lambda(x, x).$$

Proof. Let x be an arbitrary element of V, and $x = v + y$ the decomposition of x such that $v \in \operatorname{Span}(\Lambda)$ and y is perpendicular to $\operatorname{Span}(\Lambda)$. Then, by using (6) of Proposition 3.1.1, we can see that

$$\begin{aligned}a\frac{\det(\Lambda')}{\det(\Lambda)}\Delta_\Lambda(x, x) &= \det(\Lambda')a\langle y, y\rangle \\ &\geq \det(\Lambda')\langle f(y), f(y)\rangle' \\ &\geq \Delta_{\Lambda'}(f(y), f(y)).\end{aligned}$$

On the other hand, since $f(v) \in \operatorname{Span}(\Lambda')$, by (3) of Proposition 3.1.1, we can see

$$\Delta_{\Lambda'}(f(x), f(x)) = \Delta_{\Lambda'}(f(y), f(y)).$$

Thus, we get our corollary. □

3.2. **Proof.** Let us begin with the following lemma.

Lemma 3.2.1. *Let K be a finitely generated field over $\mathbb{Q}$, and A an abelian variety over K. Let Γ be a subgroup of finite rank in $A(\overline{K})$. Let X be a subvariety of $A_{\overline{K}}$, and S an infinite subset of $X(\overline{K})$ with the following properties:*

1. *S is generic, i.e., any infinite subsets of S are Zariski dense in X.*
2. *S is small with respect to $\Gamma_{div} = \{x \in A(\overline{K}) \mid nx \in \Gamma \text{ for some positive integer } n\}$.*

Then, the stabilizer of X in A is positive dimensional.

Proof. First of all, since S is infinite, $\dim(X) > 0$. We fix a positive integer n with $n > 2\dim A$. Enlarging K, we may assume that X is defined over K and there is a subgroup Γ_0 in $A(K)$ with $\Gamma_0 \subseteq \Gamma$ and $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$. By virtue of (1) of Proposition 2.4, replacing K by a finite extension field, X by $[N](X)$, and S by an infinite subset of $[N](S)$, we may assume that (S, K) is 2-minimized, where N is a positive integer. Then, by virtue of Theorem 2.5, there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_2(S, K) = C$ and $\overline{\mathcal{D}}_n(S, K) = C^{n-1}$.

If $\dim C = 0$, then every element of S is defined over K. Here we use the following well known result, which is the special case of Lang's conjecture:

> "If $X(K)$ is Zariski dense in X, then X is a translation of an abelian subvariety of A."

Thus, X is a translation of an abelian subvariety G of A. Then, $\mathrm{Stab}(X) = G$. Therefore, $\dim(\mathrm{Stab}(X)) = \dim G > 0$.

Next, we assume that $\dim(C) > 0$. Let $\pi : A \to A/C$ be the natural homomorphism, and $T = \pi(X)$. Let X^n_T be the fiber product of X over T in X^n. Then, we have a morphism $\beta_n : X^n_T \to A^{n-1}$. Since $O_K(s)^n \subseteq X^n_T$, let Y be the Zariski closure of $\bigcup_{s\in S} O_K(s)^n$ in X^n_T. Then,

$$\beta_n(Y) = \overline{\beta_n(Y)} \supseteq \overline{\beta_n\left(\bigcup_{s\in S} O_K(s)^n\right)} = C^{n-1}.$$

Therefore, we have

$$\dim(X^n_T) \geq \dim(Y) \geq \dim(C^{n-1}).$$

If the stabilizer of X is finite, then $\dim(X/T) \leq \dim(C) - 1$. Thus,

$$\begin{aligned} \dim(X^n_T) - \dim(C^{n-1}) &= (n\dim(X/T) + \dim(T)) - (n-1)\dim(C) \\ &\leq (n(\dim(C) - 1) + \dim(T)) - (n-1)\dim(C) \\ &= \dim(C) + \dim(T) - n \\ &\leq 2\dim(A) - n < 0. \end{aligned}$$

This is a contradiction. Therefore, $\dim(\mathrm{Stab}(X)) > 0$. $\square$

Let us start the proof of Theorem A. We set $\Lambda = \{\gamma_1, \ldots, \gamma_n\}$. Then, by using the height pairing

$$\langle\ ,\ \rangle^{\overline{B}}_L : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R},$$

we have the bilinear map

$$\Delta_\Lambda : A(\overline{K})_{\mathbb{R}} \times A(\overline{K})_{\mathbb{R}} \to \mathbb{R}$$

as in §§3.1. Then, $\Delta_\Lambda(x, x) = \delta^{\overline{B}}_L(\gamma_1, \ldots, \gamma_n, x)$.

Let $\mathrm{Stab}(X)$ be the stabilizer of X in A, and let $\pi : A \to A' = A/\mathrm{Stab}(X)$ be the natural morphism. We set $X' = \pi(X)$ and $\Gamma' = \pi(\Gamma)$. Then, $\mathrm{Stab}(X')$ is trivial and $\pi^{-1}(X') = X$. Let L' be a symmetric ample line bundle on A'. Then, by (2) of Proposition 1.2 (or Remark 1.3), there is a positive number a with

$$\langle \pi(x), \pi(x)\rangle^{\overline{B}}_{L'} \leq a\langle x, x\rangle^{\overline{B}}_L$$

for all $x \in A(\overline{K})$. Let $\Lambda' = \{\gamma'_1, \ldots, \gamma'_{n'}\}$ be a basis of $\Gamma' \otimes \mathbb{Q}$. Then, by Corollary 3.1.2,

$$\Delta_{\Lambda'}(\pi(x), \pi(x)) \leq a \frac{\det(\Lambda')}{\det(\Lambda)} \Delta_{\Lambda}(x, x)$$

for all $x \in A(\overline{K})$. Thus, we can see that the set $\{x' \in X'(\overline{K}) \mid \delta^{\overline{B}}_{L'}(\gamma'_1, \ldots, \gamma'_{n'}, x') \leq \epsilon\}$ is Zariski dense in X' for every positive number ϵ. Here we assume that $\dim(X') > 0$. Then, we can find a sequence $\{x'_l\}_{l=1}^{\infty}$ in $X'(\overline{K})$ with the following properties:

(1) If $l \neq m$, then $x'_l \neq x'_m$.
(2) $\{x'_l \mid l = 1, 2, \ldots\}$ is generic in X'.
(3) $\delta^{\overline{B}}_{L'}(\gamma'_1, \ldots, \gamma'_{n'}, x'_l) < 1/l$ for all l.

Here we claim the following.

Claim 3.2.1.1. $\{x'_l \mid l = 1, 2, \ldots\}$ *is small with respect to* Γ'_{div}.

In $A'(\overline{K}) \otimes \mathbb{R}$, by (6) of Proposition 3.1.1,

$$\delta^{\overline{B}}_{L'}(\gamma'_1, \ldots, \gamma'_{n'}, x'_l) = \Delta_{\Lambda'}(x'_l, x'_l) = \det(\Lambda')\langle x'_l - p_{\Lambda'}(x'_l), x'_l - p_{\Lambda'}(x'_l)\rangle^{\overline{B}}_{L'} < 1/l.$$

Here, since $\Gamma'_{\mathbb{Q}}$ is dense in $\Gamma'_{\mathbb{R}}$, there is $y'_l \in \Gamma'_{\mathbb{Q}}$ with $\det(\Lambda')\langle x'_l - y'_l, x'_l - y'_l\rangle^{\overline{B}}_{L'} < 1/l$. Since Γ'_{div} is a divisible group, y'_l comes from an element of Γ'_{div}, so that we may assume that $y_l \in \Gamma'_{div}$. Thus, if we set $z'_l = x'_l - y'_l$, then $x'_l = y'_l + z'_l$, $y'_l \in \Gamma'_{div}$, and $\det(\Lambda')\langle z'_l, z'_l\rangle^{\overline{B}}_{L'} < 1/l$. Hence $\{x'_l \mid l = 1, 2, \ldots\}$ is small with respect to Γ'_{div} by (4) of Proposition 2.1.

By this claim together with Lemma 3.2.1, we can see that $\dim(\mathrm{Stab}(X')) > 0$. This is a contradiction. Therefore, $\dim(X') = 0$, say, $X' = \{P'\}$. Then, $\Delta_{\Lambda'}(P', P') \leq \epsilon$ for every $\epsilon > 0$. Thus, $\Delta_{\Lambda'}(P', P') = 0$, which implies that $P' \in \Gamma'_{div}$. Since $\pi : \Gamma_{div} \to \Gamma'_{div}$ is surjective, there is $P \in \Gamma_{div}$ with $\pi(P) = P'$. Then, $X = \mathrm{Stab}(X) + P$. Moreover, $\mathrm{Stab}(X)$ is an abelian subvariety of A because X is a variety. Thus, we get our theorem.

Remark 3.2.2. Let K be a finitely generated field over $\mathbb{Q}$, A an abelian variety over K, and X a geometrically irreducible subvariety of A. Let

$$\langle\ ,\ \rangle^{\overline{B}}_{L} : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$$

be the height pairing associated with a big polarization $\overline{B}$ and a symmetric ample line bundle L. In the proof of this note, we used only the following two fundamental results.

- **Bogomolov's conjecture over *K***: If $\{x \in X(\overline{K}) \mid \langle x, x\rangle^{\overline{B}}_{L} \leq \epsilon\}$ is Zariski dense in X for every $\epsilon > 0$, then X is a translation of an abelian subvariety of A by a torsion point.

- **Lang's conjecture over *K* in the special case**: If $X(K)$ is Zariski dense in X, then X is a translation of an abelian subvariety of A.

Remark 3.2.3. Even in the case where K is a number field, our proof is slightly simpler than Poonen's proof. For, we avoid measure-theoretic arguments by considering a geometric trick.

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